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Delamination Analysis of Sandwich Beam: High-Order Theory

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An analytical method is presented for delamination analysis of a sandwich beam subjected to an action of a transverse load. The stress/strain state of the sandwich beam including delamination is decomposed into two states: basic state and additional state. The additional state of the delaminated sandwich beam is investigated using the theory of high-order shear strain. The ordinary differential equations for delaminated and undelaminated zones in the additional state are obtained. Because of stress singularity at delamination frontier, the usual displacement and stress continuity conditions between the delaminated and undelaminated zones are not suitable for an approximate analytical solution. Therefore, new weak form of continuity condition is suggested. Comparison with a numerical solution reveals that the analytical solution method provides good accuracy in both delaminated and undelaminated areas. A new method of calculating energy release rate of delamination growth for a sandwich beam is elaborated on the basis of the developed analytical technique. The considered sample delamination problem shows that, the higher the order of shear strain approximation is, the more precise the approximate analytical solution is.

Nomenclature

$\boldsymbol{A}$	=	$(n+1)\times(n+1)$ symmetrical matrix corresponding with normal strain energy in undelaminated area
$\tilde{\boldsymbol{A}}$	=	$n\times n$ symmetrical matrix corresponding with normal strain energy in delaminated area
a	=	half-length of the delamination area
$\boldsymbol{B}$	=	$(n+1)\times(n+1)$ symmetrical matrix corresponding with shear strain energy in undelaminated area
$\tilde{\boldsymbol{B}}$	=	$n\times n$ symmetrical matrix corresponding with shear strain energy in delaminated area
b, c	=	length of the left and right undelaminated area
$C_k, D_k, \tilde{C}_k, \tilde{D}_k$	=	arbitrary integration constants
E_c, E_f	=	Young's modulus of the sandwich core and facings
F_N, F_Q	=	total axial and transverse forces
G_c	=	shear modulus of the core
M	=	total bending moment
n	=	order of interpolating polynomial, $2m+1$
$\boldsymbol{Q}, \tilde{\boldsymbol{Q}}$	=	matrices consisting of the eigenvectors
q_k	=	kth eigenvector of matrix $\boldsymbol{A}^{-1}\boldsymbol{B}$
$\tilde{q}_k$	=	kth eigenvector of matrix $\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}$
$\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{T}}$	=	constant vectors that are relevant with the given shear stress $\bar{\tau}(x)$
$t_f, 2t_c$	=	thicknesses of the facings and the core
$u_c(x, z)$	=	axial displacement of the core at point x, z
$u_{f1}(x), u_{f2}(x)$	=	axial displacement of the upper and lower facings
$w(x)$	=	deflection of the whole beam
w_c, w_{f1}, w_{f2}	=	deflection of the core, upper facing, and lower facing
$\boldsymbol{\alpha}$	=	generalized coordinate vector of axial displacement of the core in undelaminated area, $(\alpha_0\ \alpha_1\ \cdots\ \alpha_n)^T$
$\tilde{\boldsymbol{\alpha}}$	=	generalized coordinate vector of axial displacement of the core in delaminated area, $(\alpha_0\ \alpha_1\ \cdots\ \alpha_{n-1})^T$
$\gamma_c, \tau_c, \gamma_{f1}, \tau_{f1}, \gamma_{f2}, \tau_{f2}$	=	shear strain and stress of the core and upper and lower facings
$\varepsilon_c, \sigma_c, \varepsilon_{f1}, \sigma_{f1}, \varepsilon_{f2}, \sigma_{f2}$	=	strain and stress of the core and upper and lower facings
λ_k	=	kth eigenvalue of matrix $\boldsymbol{A}^{-1}\boldsymbol{B}$
$\tilde{\lambda}_k$	=	kth eigenvalue of matrix $\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}$
ξ	=	parameter associated with comparative stiffness of the core, $E_c t_c / E_f t_f$
Π	=	total elastic energy
Π^*	=	energy release rate after crack expansion
$\bar{\tau}$	=	shear stress acting at upper and lower delaminated faces in additional state

Superscripts

L	=	left undelaminated zone
R	=	right undelaminated zone
T	=	matrix transpose
-1	=	inverse of matrix
$'$	=	first-order derivative with respect to the coordinate x
$''$	=	second-order derivative with respect to the coordinate x

I. Introduction

SANDWICH structures have many advantages. They have high specific intensity and stiffness and good thermal and acoustical insulation. As a result, they are often used in aerospace and aviation. However, one mode of failure, usually associated with this structure, is the delamination effect, which consists of separation of the facings from the core as a result of manufacturing defect or service loads. It may significantly affect the behavior of the sandwich structure through high stresses in the vicinity of delamination tip and large deformation of the overall structure.

Over the past few decades, the delamination effect in sandwich beams has been considered by many researchers.[1–5] The stress near the delamination tip and the buckling and postbuckling of the delaminated sandwich beam have been investigated in studies by means of three-dimensional numerical simulation based on the theory of

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elasticity, as well as on first-order shear strain shell theory. Frostig[6] and Frostig et al.[7] investigated the behavior of a delaminated sandwich beam with a transversely flexible core. In that research, the shear stress in the core is considered as uniform through the thickness of the core. Then Miao and Zhang[8] elaborated a simplified model of a delaminated sandwich beam. In their studies, the delamination analysis of sandwich beams was reduced to the analysis of an additional state. Such decomposition is most important for the problem analysis. The approach based on the idea of decomposition simplifies the delamination analysis, and so the decomposition technique is also used in our study. However, because of the assumption of an antiplane state,[8] the in-core transverse shear stress distribution was simplified and supposed to be uniform.[1–8] Actually our investigation showed that the distribution of shear stress along core thickness was inhomogeneous in the additional state,[9] so that the antiplane assumption cannot correctly reflect the distribution of transverse shear stress. Furthermore, the antiplane assumption ignores the in-plane stress of the core; however, our research reveals that the contribution of core-in-plane stresses to the total axial force and bending moment cannot be ignored.[9] The contribution of core-in-plane stress to axial force and bending moment is negligible only for an undelaminated sandwich beam. That is why we conducted a more accurate analysis of the additional state of a sandwich beam using the theory of high-order shear strain.

The primary goal of our research is the delamination analysis for the additional state of sandwich beam. The paper is divided into eight major sections. Section II describes the decomposition method and basic assumptions. Ordinary differential equations of the undelaminated region and their solution are obtained in Sec. III. In Sec. IV, the differential equations of the delaminated region are derived, and the solution is also obtained. In Sec. V, the boundary conditions and continuity conditions between the delaminated and undelaminated regions are discussed. Section VI provides a formula for calculating the energy release rate. The solution of a sample problem is described in Sec. VII. Distributions of normal stress and transverse shear stress in core are shown. They are compared with the finite element (FE) method solution. Finally, conclusions about the delamination analysis of a sandwich beam are given in the last section.

II. Decomposition Method and Assumptions

The sandwich beam, in which facings are symmetrically laid down on the core, is shown in Fig. 1a. The state of plane stress is assumed.

Figure 1 shows how the actual stress/strain state of the sandwich beam is decomposed into basic and additional states.

We suppose that there exists a delamination area of the length $2a$ between the facing and core that divides the beam into the delaminated and undelaminated parts. The transition area of minimum length is situated between them. In accord with the superposition principle, the actual delaminated state of the sandwich beam shown in Fig. 1a can be decomposed into the superposition of the following two states.[8] The first one is the so-called basic state (Fig. 1b). In this state, the sandwich beam does not contain delamination but is loaded by cross force and support reaction bearing. The second state is the additional one shown in Fig. 1c. The sandwich beam contains a delamination area but does not bear a cross load or support reaction. It is loaded by a couple of shearing stresses applied on the upper and lower delamination faces. These shearing stresses are equal to the shearing stresses at the same locations in the actual state multiplied by minus one.

The basic state does not refer to the delamination problem and is well studied and so we do not consider it. The additional state caused by the delamination is the key point of our research.

Our consideration is based on the following assumptions:

1) Being extremely thin, the facings are regarded as membranes, and only the in-plane stress uniformly distributed is considered. The effect of transverse shear stress is neglected.

2) The in-plane displacement of the core is approximated by an nth-order polynomial.

3) The facing and core are incompressible in the direction along the thickness.

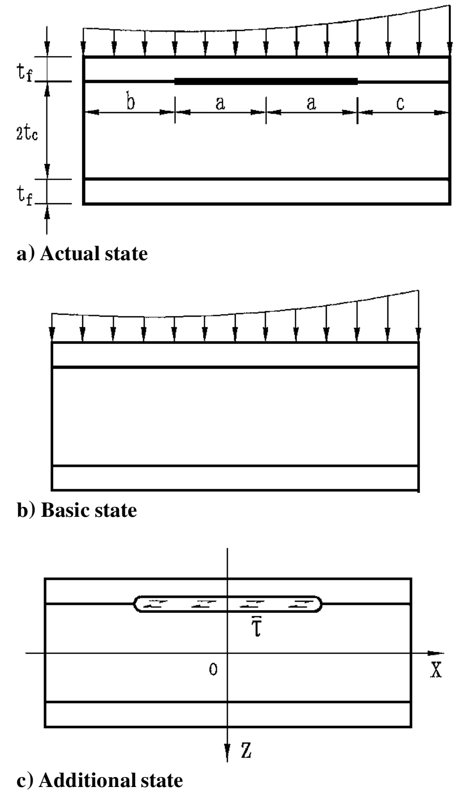


Fig. 1 Scheme of stress/strain state decomposition for sandwich beam.

According to the hypothesis, the displacements of upper and lower facings and of the core can be expressed as follows:

$$u_{f1} = u_{f1}(x), \qquad u_{f2} = u_{f2}(x), \qquad u_c = \sum_{k=0}^{n} \alpha_k(x) z^k$$

$$w_{f1} = w_{f2} = w_c = w(x) \tag{1}$$

The strain fields for each layer of the sandwich beam can be expressed as

$$\varepsilon_{f1} = u'_{f1}, \qquad \varepsilon_c = \sum_{k=0}^{n} \alpha'_k z^k, \qquad \varepsilon_{f2} = u'_{f2}$$

$$\gamma_c = w' + \sum_{k=1}^{n} k\alpha_k z^{k-1} \tag{2}$$

Using Hooke's law, we obtain

$$\sigma_{f1} = E_f \varepsilon_{f1}, \qquad \sigma_{f2} = E_f \varepsilon_{f2}, \qquad \sigma_c = E_c \varepsilon_c$$

$$\tau_c = G_c \left[w'(x) + \frac{\partial u_c}{\partial z} \right] \tag{3}$$

Assumption 3 leads to the equalities

$$\tau_{f1} = \tau_{f2} = 0 \tag{4}$$

Because the additional state is a self-equilibrium one, the total axial force, shear force, and bending moment at an arbitrary cross section (including the delamination sector) should equal zero, that is,

$$F_N = \sigma_{f1} t_f + \int_{-t_c}^{t_c} \sigma_c \, \mathrm{d}z + \sigma_{f2} t_f = 0 \tag{5a}$$

$$M = -\sigma_{f1} t_f t_c + \int_{-t_c}^{t_c} \sigma_c z \, \mathrm{d}z + \sigma_{f2} t_f t_c = 0 \tag{5b}$$

$$F_Q = \int_{-t_c}^{t_c} \tau_c \, \mathrm{d}z = 0 \tag{5c}$$

From Eqs. (2) and (3), Eq. (5a) can be expressed as

$$F_Q = 2G_c t_c \left[w' + \alpha_1 + \sum_{i=1}^{(n-1)/2} (2i+1) t_c^{2i} \alpha_{2i+1} \right] = 0 \quad (6)$$

When w' is removed from Eq. (6), the shear strain of the core may be written as

$$\gamma_c = \sum_{i=1}^{(n-1)/2} 2i\alpha_{2i} z^{2i-1} + \sum_{i=1}^{(n-1)/2} \left[(2i+1) z^{2i} - t_c^{2i} \right] \alpha_{2i+1} \quad (7)$$

III. Governing Equation of the Undelamination Sector

For example, we consider the right undelaminated sector. In the undelaminated sector, there exist the following displacement continuity conditions between the upper and lower facings and the core:

$$u_{f1}(x) = u_c(x, -t_c), \qquad u_{f2}(x) = u_c(x, t_c) \quad (8)$$

For the sake of simplicity, we regard the boundary $x = a$ between the delamination and undelamination sectors as the boundary where displacement is given. (This supposition does not affect the obtained differential equation.) The total elastic energy of the right undelamination zone can be expressed as

$$\Pi = \int_a^{a+c} \frac{1}{2} \left[\sigma_{f1} \varepsilon_{f1} t_f + \int_{-t_c}^{t_c} (\sigma_c \varepsilon_c + \tau_c \gamma_c) \, \mathrm{d}z + \sigma_{f2} \varepsilon_{f2} t_f \right] \mathrm{d}x \quad (9)$$

Then when Eqs. (2), (3), and (7) are used, the total potential energy of the system can be reduced to

$$\Pi = \frac{1}{2} \int_a^{a+c} [(\boldsymbol{\alpha}')^T \boldsymbol{A} \boldsymbol{\alpha}' + \boldsymbol{\alpha}^T \boldsymbol{B} \boldsymbol{\alpha}] \, \mathrm{d}x \quad (10)$$

where $\boldsymbol{A}$ is positive definite and $\boldsymbol{B}$ is positive semidefinite and they correspond to the normal stress energy and shear stress energy, respectively. The elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ may be expressed as

$$A_{ij} = E_f t_f [1 + (-1)^{i+j}] \left(1 + \frac{\xi}{1+i+j} \right) \qquad (0 \le i, j \le n) \quad (11a)$$

$$B_{ij} = \begin{cases} 0 & (0 \le i, j \le 1, \text{ or } i+j \text{ is odd number}) \\ \dfrac{G_c t_c i j}{i+j+1} & (2 \le i, j \le n, \text{ and } i, j \text{ both are even numbers}) \\ \dfrac{G_c t_c (ij - i - j - 1)}{i+j+1} & (2 \le i, j \le n, \text{ and } i, j \text{ both odd numbers}) \end{cases} \quad (11b)$$

In the preceding formula, we introduce parameter ξ, which accounts for the contribution of the core layer to the whole stiffness.

When the variation of Eq. (10) is taken,

$$\delta\Pi = \int_a^{a+c} [(\boldsymbol{\alpha}')^T \boldsymbol{A} \delta\boldsymbol{\alpha}' + \boldsymbol{\alpha}^T \boldsymbol{B} \delta\boldsymbol{\alpha}] \, \mathrm{d}x$$

$$= (\boldsymbol{\alpha}')^T \boldsymbol{A} \delta\boldsymbol{\alpha} |_a^{a+c} + \int_a^{a+c} [-(\boldsymbol{\alpha}'')^T \boldsymbol{A} + \boldsymbol{\alpha}^T \boldsymbol{B}] \delta\boldsymbol{\alpha} \, \mathrm{d}x \quad (12)$$

Also taking into account the arbitrariness of $\delta\boldsymbol{\alpha}$, we obtained

$$\boldsymbol{A}\boldsymbol{\alpha}'' = \boldsymbol{B}\boldsymbol{\alpha} \quad (13)$$

Let the eigenvalues and eigenvectors of matrix $\boldsymbol{A}^{-1}\boldsymbol{B}$ be λ_k and $\boldsymbol{q}_k$, $k = 0, 1, \ldots, n$, corresponding,

$$\boldsymbol{Q} = (\boldsymbol{q}_0 \quad \boldsymbol{q}_1 \quad \cdots \quad \boldsymbol{q}_n), \qquad \Lambda = \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

then

$$\boldsymbol{A}^{-1}\boldsymbol{B} = \boldsymbol{Q} \Lambda \boldsymbol{Q}^{-1} \quad (14)$$

Introducing one more notation,

$$\boldsymbol{\beta} = \boldsymbol{Q}^{-1} \boldsymbol{\alpha} \quad (15)$$

we can reduce Eq. (13) to the following system of decoupled ordinary differential equations:

$$\beta_k'' - \lambda_k \beta_k = 0, \qquad k = 0, 1, \ldots, n \quad (16)$$

where β_k is the kth element of vector $\boldsymbol{\beta}$. Because γ_c is independent of α_0 and α_1 in Eq. (7), matrix $\boldsymbol{B}$ in Eq. (11b) has two zero rows and zero columns, and the submatrix obtained by eliminating two zero rows and zero columns of $\boldsymbol{B}$ is positive because it represents shear stress energy. Thus we can conclude that matrix $\boldsymbol{A}^{-1}\boldsymbol{B}$ has only two zero eigenvalues, which we denote $\lambda_{n-1} = \lambda_n = 0$.

Then the general solution of Eq. (16) is as follows:

$$\beta_k = C_k e^{\sqrt{\lambda_k} x} + D_k e^{-\sqrt{\lambda_k} x}, \qquad k = 0, 1, \ldots, n-2$$

$$\beta_k = C_k x + D_k, \qquad k = n-1, n \quad (17)$$

After arbitrary constants C_k and D_k are determined, we can find vector $\boldsymbol{\alpha}$ from Eq. (15).

IV. Governing Equation of the Delamination Sector

Within the delamination sector, the lower facing and the core are endued with the following displacement continuity condition:

$$u_{f2}(x) = u_c(x, t_c) \quad (18)$$

First, we consider the potential energy of the core layer and lower facing, which can be expressed as

$$\Pi = \int_{-a}^{a} \frac{1}{2} \left[\int_{-t_c}^{t_c} (\sigma_c \varepsilon_c + \tau_c \gamma_c) \, \mathrm{d}z + \sigma_{f2} \varepsilon_{f2} t_f \right] \mathrm{d}x$$

$$- \int_{-a}^{a} -\bar{\tau} u_c(x, -t_c) \, \mathrm{d}x \quad (19)$$

where the last term is the potential of external force, caused by given shear stress $\bar{\tau}$ applied on the surface of delamination. Here we do not consider the potential energy of external force at the sections $x = \pm a$. Because $\bar{\tau}(x)$ is given, from Eq. (7) we obtained

$$\alpha_n = \frac{1}{n-1} \left[\frac{\bar{\tau}(x)}{2G_c t_c^{n-1}} + \sum_{i=1}^{(n-1)/2} 2i\alpha_{2i} t_c^{2i-1} \right]$$

$$- \frac{1}{n-1} \sum_{i=1}^{(n-1)/2-1} 2i t_c^{2i} \alpha_{2i+1} \quad (20)$$

Taking the variation of Eq. (19) and also utilizing Eqs. (2), (3), (7), and (20), we obtained

$$\delta\Pi = \int_{-a}^{a} [\delta(\tilde{\alpha}')^T (\tilde{\boldsymbol{A}}\tilde{\alpha}' - \bar{\tau}'\tilde{S}) + \delta\tilde{\boldsymbol{\alpha}}^T (\tilde{\boldsymbol{B}}\tilde{\alpha} - \bar{\tau}\tilde{\boldsymbol{T}})] \, \mathrm{d}x \quad (21)$$

where $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ are, correspondingly, the positive definite and positive semidefinite symmetric matrices. These matrices correspond to the normal and shear stress energies of the delamination sector, respectively. Vectors $\tilde{\boldsymbol{S}}$ and $\tilde{\boldsymbol{T}}$ are constant, with which the given shear stress $\bar{\tau}(x)$ is relevant. To obtain expressions of all of these matrices, we used commercial software Mathematica. Taking the integration of the first term of the preceding equation by parts and taking into account the arbitrariness of $\delta\boldsymbol{\alpha}$, we obtained

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{\alpha}}'' - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{\alpha}} = \tilde{\boldsymbol{S}}\bar{\tau}'' - \tilde{\boldsymbol{T}}\bar{\tau} \quad (22)$$

Let the eigenvalues of matrix $\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}$ be $\tilde{\lambda}_k$, $k=0,1,n-1$, and the matrix $\tilde{\boldsymbol{Q}}$ be composed of the correspondent eigenvectors. By the use of the notation

$$\tilde{\beta} = \tilde{\boldsymbol{Q}}^{-1}\tilde{\alpha}, \qquad \tilde{\boldsymbol{F}} = \tilde{\boldsymbol{Q}}^{-1}\tilde{\boldsymbol{A}}^{-1}(\tilde{\boldsymbol{S}}\bar{\tau}'' - \tilde{\boldsymbol{T}}\bar{\tau}) \tag{23}$$

Eq. (22) can be decoupled as follows:

$$\tilde{\beta}_k'' - \tilde{\lambda}_k\tilde{\beta}_k = \tilde{F}_k, \qquad k = 0, 1, \ldots, n-1 \tag{24}$$

where $\tilde{\beta}_k$ is the kth element of vector $\tilde{\beta}$ and $\tilde{F}_k$ is the kth element of vector $\tilde{F}$. Like matrix $\boldsymbol{A}^{-1}\boldsymbol{B}$ in the preceding section, matrix $\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}$ only has two zero eigenvalues, $\tilde{\lambda}_{n-2} = \tilde{\lambda}_{n-1} = 0$. Then the solution of Eq. (24) can be written as

$$\tilde{\beta}_k = \tilde{C}_k e^{\sqrt{\tilde{\lambda}_k}x} + D_k e^{-\sqrt{\tilde{\lambda}_k}x} + g_k(x), \qquad k = 0, 1, \ldots, n-3$$

$$\tilde{\beta}_k = \tilde{C}_k x + \tilde{D}_k, \qquad k = n-2, n-1 \tag{25}$$

where $\tilde{C}_k$ and $\tilde{D}_k$ are arbitrary constants and $\tilde{g}_k(x)$ is the particular solution. We can determine $\tilde{\alpha}_k$, $k = 0, 1, \ldots, n-1$, from formula (23).

The governing equation of the upper facing in the delamination area is as follows:

$$E_f u_{f1}'' + \bar{\tau}(x) = 0 \tag{26}$$

It has the following solution:

$$u_{f1} = \tilde{C}_n + \tilde{D}_n x + \tilde{g}_n(x) \tag{27}$$

where $\tilde{C}_n$ and $\tilde{D}_n$ are arbitrary constants and $\tilde{g}_n(x)$ is a particular solution.

V. Boundary and Continuity Conditions

First, we consider the boundary conditions. The solutions obtained in the preceding sections have $6(n+1)$ unknown constants. We only consider the case that $b \gg a$ and $c \gg a$. To ensure the stresses at section $x = -b-a$ to be zero, such as $\sigma_{f1}|_{x=-b-a} = 0$, $\sigma_c|_{x=-b-a} = 0$, etc., we should take $D_k^L = 0$, $k = 0, 1, \ldots, n-1$, and $C_n^L = 0$ in Eq. (17). In the same way, to ensure the stresses at $x = a + c$ to be zero, such as $\sigma_{f1}|_{x=a+c} = 0$, $\sigma_c|_{x=a+c} = 0$, etc., we should take $C_k^R = 0$, $k = 0, 1, \ldots, n$, in Eq. (17).

Within three sectors, Eqs. (5) have to be satisfied, and Eq. (5c) has been used in the derivation. Equations (5a) and (5b) can be used as the boundary conditions, which has six conditions in all.

Considering solution compatibility condition for the section $x = \pm a$, we can find that there are four conditions that represent displacement and stress continuity of the upper facing. However, so far we lack as many as $4n-6$ conditions.

Second, it is well known that there is stress singularity at the delamination tip $(\pm a, -t_c)$. Because of stress concentration, it is difficult to approximate accurately the displacement and stress fields in the vicinity of the tip by means of an nth-order polynomial. Therefore, it is not reasonable to satisfy stress and displacement continuity conditions on these sections. Therefore, we impose the continuity condition in a weak form. It is obvious that the area of high stress is small and is located around the delamination tip. Thus, we impose stress and displacement continuity conditions in the two sections under consideration only in $n-1$ points z_k inside the interval $-\delta t_c \le z \le t_c$, where $0 < \delta < 1$ is a parameter. We introduce this parameter to maximize the solution accuracy. It depends on the polynomial order n, and the greater n is, the greater δ should be. Thus, the continuity conditions have the following form

$$\sigma_c(\pm a^-, z_k) = \sigma_c(\pm a^+, z_k), \qquad u_c(\pm a^-, z_k) = u_c(\pm a^+, z_k) \tag{28}$$

where

$$z_k = -t_c + [(k-1)/(n-2)](1+\delta)t_c, \qquad k = 1, 2, \ldots, n-1$$

Thus, there are $4n-4$ additional conditions in the section $x = \pm a$ and only $4n-6$ unknowns and so we have to use a least-squares method to find the solution.

VI. Calculating the Formula of Energy Release Rate

Utilizing the analytical solution obtained in the preceding section, we give a new method for calculating the energy release rate of the actual state.

We denote the stress components, the strain components, and the deflection of basic state (Fig. 1b) as σ_{ij}^0, ε_{ij}^0, and w^0 and the stress components, the strain components, and the deflection of additional state (Fig. 1c) as σ_{ij}, ε_{ij}, and w. Therefore, the total elastic energy of the actual state can be expressed as follows:

$$\Pi = \frac{1}{2}\int_{-a-b}^{a+c}\int_{-t_c-t_f}^{t_c+t_f}\left(\sigma_{ij} + \sigma_{ij}^0\right)\left(\varepsilon_{ij} + \varepsilon_{ij}^0\right)\mathrm{d}z\,\mathrm{d}x - \int_{-a-b}^{a+c} q(w + w^0)\,\mathrm{d}x \tag{29}$$

where q is transverse loads. W_{12} is the virtual work of the stress taken from the basic state and strain taken from the additional state; vice versa, W_{21} is the virtual work of stress taken from the additional state and strain taken from the basic state:

$$W_{12} = \int_{-a-b}^{a+c}\int_{-t_c-t_f}^{t_c+t_f}\sigma_{ij}^0\varepsilon_{ij}\,\mathrm{d}z\,\mathrm{d}x$$

$$W_{21} = \int_{-a-b}^{a+c}\int_{-t_c-t_f}^{t_c+t_f}\sigma_{ij}\varepsilon_{ij}^0\,\mathrm{d}z\,\mathrm{d}x \tag{30}$$

The reciprocal theorem of elasticity states that both works are equal to each other:

$$W_{12} = W_{21} \tag{31}$$

The external force in the additional state is caused by shear stress that is equal in magnitude and has opposite direction to the shear stress in the basic state. Thus, we obtain

$$\int_{-a-b}^{a+c}\int_{-t_c-t_f}^{t_c+t_f}\sigma_{ij}\varepsilon_{ij}^0\,\mathrm{d}z\,\mathrm{d}x = \int_{-a-b}^{a+c}\int_{-t_c-t_f}^{t_c+t_f}\sigma_{ij}^0\varepsilon_{ij}\,\mathrm{d}z\,\mathrm{d}x = 0 \tag{32}$$

Then Eq. (29) can be reduced as follows:

$$\Pi = \frac{1}{2}\int_{-a-b}^{a+c}\int_{-t_c-t_f}^{t_c+t_f}\left(\sigma_{ij}^0\varepsilon_{ij}^0 + \sigma_{ij}\varepsilon_{ij}\right)\mathrm{d}z\,\mathrm{d}x - \int_{-a-b}^{a+c} q(w + w^0)\,\mathrm{d}x \tag{33}$$

Suppose that the crack grows da in the right direction. Taking into account that the basic state is independent of crack length and using Eq. (33), we can write the expression for the energy release rate after crack expansion as follows:

$$\Pi^* = -\frac{\mathrm{d}\Pi}{\mathrm{d}a} \tag{34}$$

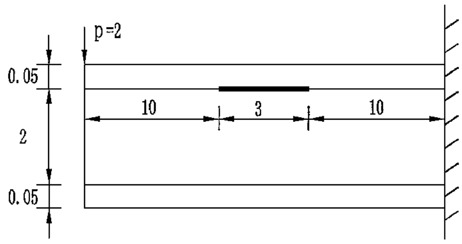
or

$$\Pi^* = -\frac{\mathrm{d}}{\mathrm{d}a}\int_{-a-b}^{a+c}\left(\int_{-t_c-t_f}^{t_c+t_f}\frac{1}{2}\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}z - qw\right)\mathrm{d}x \tag{35}$$

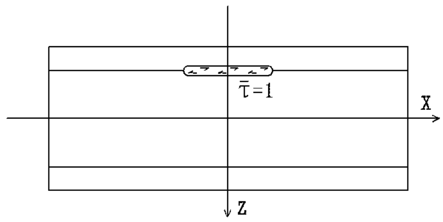
Therefore, the formula for the energy release rate formally uses only an analytical solution obtained for the additional state. Indeed, this solution utilizes the shear stress $-\bar{\tau}$ taken from the solution for the basic state.

VII. Analysis of Actual Example

As an example, we analyze the sandwich beam shown in Fig. 2a. Dimensionless quantities are adopted in this example: $E_c = 1$, $G_c = \frac{3}{8}$, $t_c = 1$, $a = 1.5$, $t_f = 0.05$, $E_f = 1000$, $p = 2$, and $b = 10$. In the basic state, the sandwich beam does not contain delamination but is loaded by cross force and a support reaction bearing. Known from the Allen model,[10] the distribution of shear stress in



a) Actual state



b) Additional state

Fig. 2 Sample sandwich beam.

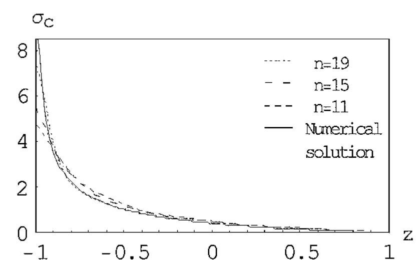


Fig. 3 Additional stress in core at cross section $x = 1.44$.

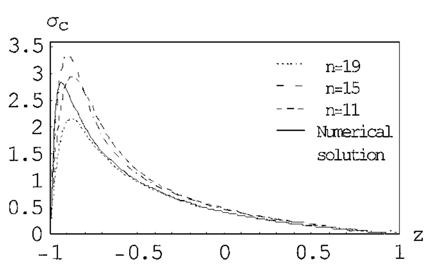


Fig. 4 Additional stress in core at cross section $x = 1.56$.

the core is uniform in the basic state according to the antiplane assumption, the facings are regarded as membranes, only the in-plane stress uniformly distributed is considered, and the effect of transverse shear stress is neglected. Thus, we take $\tau^0 = -p/(2^* t_c) = -1$ on the delaminated surface in the basic state so that $\bar{\tau} = -\tau^0 = 1$ in the additional state (Fig. 2b). Therefore, the additional state is antisymmetrical, and we can consider only one-half of the whole domain, for example, the right half.

We used several values of n to obtain the solution, for example, $n = 11$, 15, and 19 and compared it with the FE solution (solid line) based on the equations of plane elasticity. The normal stress distribution at the cross section is shown in Figs. 3–5 in the vicinity of the cross section $x = a$. The accuracy of the solution is heightened gradually when n increases. Also the accuracy is better for sections that are far apart from the section $x = a$. We notice that the analytical solution has good accuracy compared with the numerical solution. Only within a small area in the vicinity of section $x = a$ (called a transition sector in this paper), the analytical solution has low accuracy. However, if the order of polynomial increases, the transition area reduces.

Second, despite that the stress on the border between the delaminated zone and undelaminated one is not continuous, it coincides with the numerical solution well enough. That proves that the continuity conditions suggested in the paper are reasonable. The shear stress distribution along cross sections is shown in Figs. 6–8. Notice that the accuracy of the shear stress is less than the accuracy of

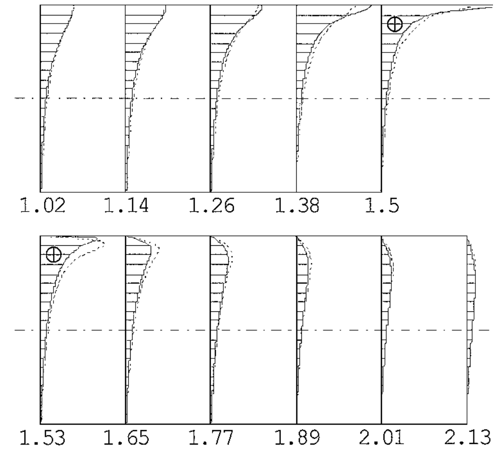


Fig. 5 Additional normal stress in core at cross section near tip ($n = 15$).

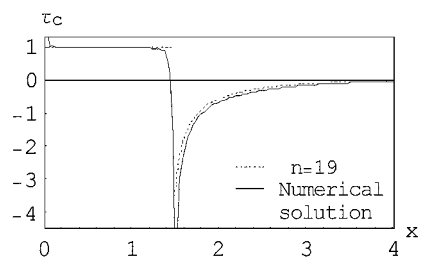


Fig. 6 Additional shear stress in core at section $z = -t_c$.

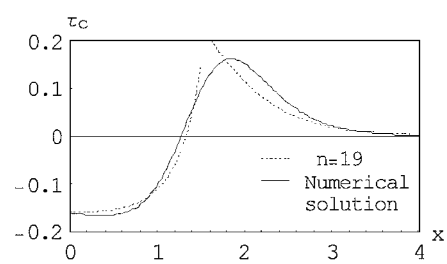


Fig. 7 Additional shear stress in core at section $z = 0$.

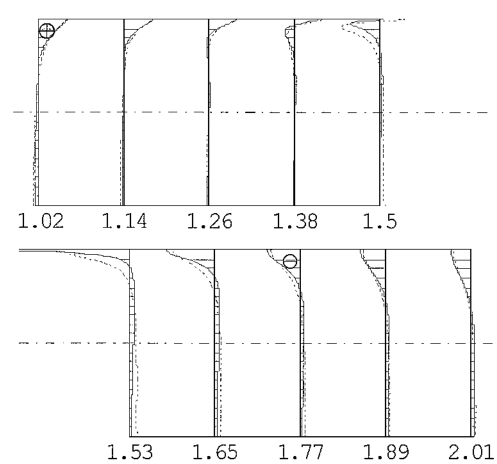


Fig. 8 Additional shear stress in core at cross section near tip ($n = 19$).

normal stress for the same order of polynomial. The deflection at the center is shown in Fig. 9. When the order of the polynomial equals 19, the analytical solution almost coincides with numerical solution.

Third, Table 1 presents the maximum relative error of normal stress in the core computed using the analytical solution. The numerical solution was computed using a fine mesh and so we consider it as precise. Because there are no other analytical solutions, we could not use any other way to verify our solution.

The numerical solution was computed using plane elements. It is compared with the analytical solution at sections with axial

Table 1 Maximum relative error of normal stress in the core

	Relative error, %				
n	$x = 1.02$	$x = 1.14$	$x = 1.26$	$x = 1.38$	$x = 1.50$
7	18.5	23.3	30.7	42.3	61.1
13	6.9	9.9	14.9	23.7	40.4
19	0.4	1.8	4.5	9.6	20.0

Table 2 Maximum relative error of deflection

n	Error, %
7	28.3
11	13.2
13	10.5
15	7.5
19	2.9

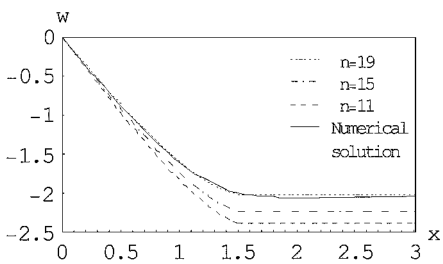


Fig. 9 Deflection relative the center of delamination.

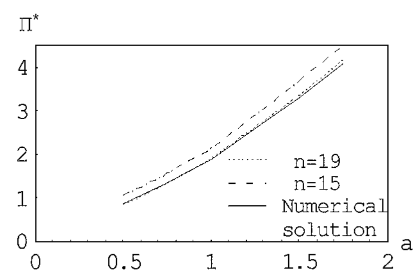


Fig. 10 Energy release rate vs delamination length.

coordinates $x = 1.02$, 1.14, 1.26, 1.38, and 1.50 for $n = 7$, 13, and 19. In the same way, Table 2 gives the maximum relative error of deflection for $n = 7, 11, 13, 15$, and 19. Tables 1 and 2 confirm the conclusions on the analytical solution accuracy given in earlier sections.

Finally, the change of energy release rate in the actual state when the delamination length a grows is shown in Fig. 10. Again one can see that the analytical solution approaches the numerical one when the order of polynomial increases.

VIII. Conclusions

A method of delamination analysis of sandwich beams based on higher-order shear deformation theory is elaborated. To obtain a solution approximated by a polynomial of high order, the weak form of the continuity condition on the boundary between the delaminated and undelaminated zones is suggested. Computations proved that such a condition is effective to achieve good accuracy. For sufficient accuracy, a polynomial approximation of high order is also required. A new method to calculate energy release rate of delamination growth of sandwich beam is developed. It provides a powerful tool for fracture analysis of sandwich beam.

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